REFERENCES

1. Neustadt, L. W., Time-optimal control systems with position and integral limits. J. Math. Analysis and Applications, Vol. 3, №3, 1961.
2. Singh, R. N. P., Functional analysis approach to optimal control problems with multiple constraints on the controlling function. Internat. J. Control, Vol. 9, №1, 1969.
3. Formal'skii, A. M., The time-optimal control problem in systems with controlling forces of bounded magnitude and impulse. PMM Vol. 34, №5, 1970.
4. Fel'dbaum, A. A., Fundamentals of the Theory of Optimal Automatic Systems. Moscow, Fizmatgiz, 1963.
5. Boltianskii, V. G., Mathematical Methods of Optimal Control. Moscow, "Nauka", 1969.
6. Formal'skii, A. M., Controllability region of systems with constrained control resources. Avtomatika i Telemekhanika, №3, 1968.
7. Shilov, G. E., Mathematical Analysis, Pergamon Press (Translation from Russian), Book №10796, 1965.
8. Kalman, R. E., On the general theory of control systems. Proc. First IFAC Congress, Vol. I, London, Butterworths, 1960.
9. Kreindler, E., Contributions to the theory of time-optimal control. J. of Franklin Institute, Vol. 275, №4, 1963.
10. Neustadt, L. W., Optimization a moment problem and nonlinear programing. SIAM J. on Control, Vol. 2, №1, 1964.
11. Markhashov, L. M., Time-optimal pulse operation in linear systems. PMM Vol. 32, №1, 1968.
12. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1968.

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HYDRODYNAMIC INTERACTION BETWEEN BODIES IN A PERFECT INCOMPRESSIBLE FLUID AND THEIR MOTION IN NONUNIFORM STREAMS

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A general method based on the use of Lagrangian equations for determining hydrodynamic interaction between bodies in a fluid is presented. Formulas for the kinetic energy and the Lagrangian function are reduced to a form which permits an effective application of the method of small parameter. Additive components of kinetic energy and of the Lagrangian function, which determine the hydrodynamic interaction between two bodies, one of which is small in comparison with the distance between the two, are calculated. The method is used for considering the case of several bodies. The results are expressed in terms of coefficients of apparent mass of individual bodies in a boundless fluid. General

formulas are derived for forces and moments acting on a body in a nonuniform stream.

The feasibility of expressing hydrodynamic reactions on a body in a nonuniform stream in terms of apparent mass coefficients has not been, so far, established. Solutions were sought for bodies of particular form, while in the three-dimensional case no solution was found even for a small sphere.

The description of a solid body in a perfect incompressible fluid by Lagrangian equations was first given in [1]. The forces and moments acting on such body moving in a boundless fluid were determined in [2]. The most rigorous proof of equations of motion of a solid body in a boundless fluid appears in [3]. The problem of a body in an arbitrary stream of fluid was apparently first formulated by Zhukovskii [4]. The motion of a body in a uniform accelerated stream was investigated in [5]. Formulas for forces and moments acting on a stationary elliptical cylinder in an arbitrary plane potential flow were derived in [6], while in [7] the force acting on an expanding circular cylinder moving in an arbitrary stream with constant vorticity was calculated. As survey of publications on the hydrodynamic interaction between bodies in a fluid appears in [8].

1. Energy of hydrodynamic interaction. Expressions defining forces exerted by the fluid on a body and, consequently, the equations of motion of the body in a fluid can be derived by solving the problem of Laplacian equation with subsequent application of the Cauchy-Lagrange integral. However such calculations in the case of a nonuniform stream, where the presence of a velocity gradient is important, is extremely cumbersome even for a sphere. To avoid solving in each particular case the Lapalcian equation an attempt can be made to use the Lagrangian equation. As shown in [9 – 11], the motion of solid bodies in a perfect incompressible fluid is defined by Lagrangian equations. With the use of the variational principle [12] we can prove that Lagrangian equations are valid in the general case of motion of a solid body in a perfect incompressible fluid, when the potential flow in which the body is immersed is generated by some arbitrary motion of surfaces subject to kinematic or dynamic conditions.

Let a body whose surface is S_1 move in a potential flow of a perfect incompressible fluid which is at rest at infinity. The nonuniform stream is induced by the motion of surface S_2 (possibly consisting of several interconnected parts) along which kinematic (fixed law of motion) and dynamic (free surface) conditions are specified. To formulate the Lagrangian equations with respect to coordinates q_{1j} of surface S_1 it is necessary to separate the components of the fluid kinetic energy, which depend on these coordinates and velocities $q_{1j}^{\cdot}$ associated with these coordinates. The complete definition of the motion requires, generally speaking, the indication of the set of coordinates q_{2j} specifying surface S_2 and velocities $q_{2j}^{\cdot}$. Let Φ_α denote the velocity potential of motion S_α at velocities $q_{\alpha j}^{\cdot}$ in the boundless fluid ($\alpha = 1, 2$), and Φ the general flow potential in the presence of the two surfaces. In this notation there exists a unique representation of Φ in the form of the sum

$$\Phi = \Phi_1 + \Phi_1' + \Phi_2 + \Phi_2' \tag{1.1}$$

Outside S_α ($\alpha = 1, 2$) function Φ_α' is harmonic and tends to vanish at infinity, since by virtue of the fundamental identity of the theory of harmonic functions we have

$$\Phi_\alpha + \Phi_\alpha' = \frac{1}{4\pi} \int\limits_{S_\alpha} \left(\frac{1}{r} \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n} \frac{1}{r} \right) dS \tag{1.2}$$

which proves the validity of formula (1.1). Here and subsequently differentiation is carried out along the external normal to the fluid volume. Formula (1 2) is useful for estimating the potential at considerable distances from S_α. At surface S_α potential Φ satisfies the following condition:

$$\partial \Phi \,/\, \partial n = \partial \Phi_\alpha \,/\, \partial n \tag{1.3}$$

The kinetic energy of fluid is

$$T = T_1 + T_{12} + T_2 = \frac{\rho}{2} \int\limits_{S_1 \cup S_2} \Phi \frac{\partial \Phi}{\partial n}\, dS \tag{1.4}$$

where T_α is the fluid energy when one surface S_α is in motion in a boundless fluid, and T_{12} is the energy of hydrodynamic interaction

$$T_\alpha = \frac{\rho}{2} \int\limits_{S_\alpha} \Phi_\alpha \frac{\partial \Phi_\alpha}{\partial n}\, dS \tag{1.5}$$

The determination of the hydrodynamic interaction T_{12} is equivalent to the determination of kinetic energy variation $\Delta T = T_1 + T_{12}$, when the first body is immersed in the stream induced by the second body, since T_1 is known from the problem of motion of a body in a boundless fluid. The variation of kinetic energy may be considered in a more general case in which the potentiality condition is imposed on the unperturbed stream $\mathbf{v}_2$, while the shape and volume of the body may vary.

Theorem. Let the potential stream $\mathbf{v}_2 = \nabla \Phi_2$ be specified in the region bounded by S_2 with $\Phi_2 \to 0$ at infinity. By immersing in this stream a body with an arbitrarily moving boundary S_1, with constant normal velocity v_n at surface S_2, a velocity field $\mathbf{v}$ is generated which is, generally speaking, a vortical field. In this case the kinetic energy variation ΔT is defined by formula

$$\frac{\Delta T}{\rho} = \frac{1}{2} \int\limits_{\Omega} (\mathbf{v} - \mathbf{v}_2)^2 d\tau + \int\limits_{V} \left(\frac{\partial \Phi_2}{\partial t} + \frac{\mathbf{v}_2{}^2}{2} \right) d\tau - \frac{d}{dt} \int\limits_{V} \Phi_2 d\tau \tag{1.6}$$

where Ω is the region occupied by the fluid in the presence of the body, and V is the volume of the body bounded by surface S_1.

Proof. By subtracting the kinetic energy of the stream in region $\Omega \cup V$ prior to the immersion of the body from the kinetic energy of the perturbed stream, we can obtain

$$\frac{2}{\rho} \Delta T = \int\limits_{\Omega} (\mathbf{v} - \mathbf{v}_2)^2 d\tau + 2 \int\limits_{\Omega} (\mathbf{v} - \mathbf{v}_2)\, \mathbf{v}_2 d\tau - \int\limits_{V} \mathbf{v}_2{}^2 d\tau$$

Taking into account the solenoidal property of $\mathbf{v}$ and that of potentiality of $\mathbf{v}_2$, and using Ostrogradskii's theorem, we reduce the second integral to the integral of $(v_n - v_{2n})\, \Phi_2$ taken over the surface S_1 of the body, because the integral over S_2 vanishes owing to the theorem condition that $v_n = v_{2n}$ at S_2. The integral of $-v_{2n}\Phi_2$ over surface S_1 is equal to the integral of $\mathbf{v}_2{}^2$ taken over volume V. Finally, we obtain

$$2 \int\limits_{\Omega} (\mathbf{v} - \mathbf{v}_2)\, \mathbf{v}_2 d\tau - \int\limits_{V} \mathbf{v}_2{}^2 d\tau = 2 \int\limits_{S_1} v_n \Phi_2 dS + \int\limits_{V} \mathbf{v}_2{}^2 d\tau$$

The substitution of the last formula into the preceding one with allowance for the kinematic identity

$$\frac{d}{dt}\int\limits_V \Phi_2 d\tau = \int\limits_V \frac{\partial \Phi_2}{\partial t} d\tau - \int\limits_{S_1} v_n \Phi_2 dS$$

for the total derivative of the integral over the moving volume, leads to formula (1.6).

It should be noted that this proof is based on the assumption that the velocity $\mathbf{v}$ diminishes fairly rapidly at infinity, an assumption which is necessary for the existence of kinetic energy.

For a given unperturbed stream the last two integrals in (1.6) are functions of coordinates and velocities of the body. For a fairly small body the first integral in (1.6) is approximately equal to the kinetic energy of the fluid in relative motion.

If the velocity field $\mathbf{v}$ is potential and the potential of mass forces is denoted by U, we can introduce the Lagrangian function

$$L = \Delta T - \rho \int\limits_V U d\tau$$

which depends on generalized coordinates and velocities of surface S_1 and is related to generalized forces exerted on the fluid by surface S_1 by Lagrangian equations.

Expressing the integrand of the second integral in (1.6) in terms pf pressure p_2 of the unperturbed stream with the use of the Cauchy-Lagarange integral, and discarding the total derivative, which is unessential for the Lagrangian function, we can write for the Lagrangian function related to the motion of S_1 the following exact expression:

$$L = \frac{\rho}{2}\int\limits_\Omega (\mathbf{v} - \mathbf{v}_2)^2 d\tau - \int\limits_V p_2 d\tau \qquad (1.7)$$

2. The case of a solid body. Let a solid body be bounded by surface S_1. The velocity field potential can be then represented in the form [9]:

$$\Phi_1(\mathbf{q}^{\cdot}, \boldsymbol{\omega}, \mathbf{x}) = U_\alpha \varphi_\alpha, \qquad U_i = q_i^{\cdot}, \qquad U_{i+3} = \omega_i \qquad (2.1)$$
$$(\alpha = 1, 2, \ldots, 6;\ i = 1, 2, 3)$$

Here and subsequently x_i are Cartesian coordinates and ω_i are components of the body angular velocity. Recurrent subscripts denote summation. Functions φ_α are harmonic which decrease toward infinity and satisfy on S_1 the following boundary conditions:

$$\frac{\partial \varphi_i}{\partial n} = n_i, \quad \frac{\partial \varphi_{i+3}}{\partial n} = \varepsilon_{ijk}\Delta x_j n_k, \quad \Delta x_j = x_j - q_j \qquad (2.2)$$

where ε_{ijk} is the Levi-Civita tensor.

We represent the kinetic energy T_1 in terms of the following quadratic form:

$$T_1(\mathbf{q}^{\cdot}, \boldsymbol{\omega}) = \frac{\rho}{2}\lambda_{\alpha\beta}U_\alpha U_\beta, \qquad \lambda_{\alpha\beta} = \int\limits_{S_1} \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n} dS \qquad (2.3)$$
$$(\alpha, \beta = 1, 2, \ldots, 6)$$

The apparent masses (2.3) for $\alpha, \beta > 3$ are expressed in terms of the antisymmetric part of tensor

$$\Gamma_{ijk} = \int\limits_{S_1} \varphi_i \Delta x_j n_k dS \qquad (2.4)$$

$$\lambda_{ij+3} = \lambda_{j+3i} = \varepsilon_{jlk}\Gamma_{ilk}$$

Let D be the maximum distance between points of surface S_1.. At the surface of the body $\varphi_i \sim D$ and $\varphi_{i+3} \sim D^2$, hence from (2.3) we have

$$\lambda_{ij} \sim D^3, \quad \lambda_{ij+3} \sim D^4, \quad \Gamma_{ijk} \sim D^4, \quad \lambda_{i+3j+3} \sim D^5$$

To calculate the interaction energy by (1.6) or the Lagrangian function (1.7) it is necessary to determine the integral

$$\int_{\Omega} (\mathbf{v} - \mathbf{v}_2)^2 d\tau = \int_{S_1} (\Phi - \Phi_2) \frac{\partial}{\partial n} (\Phi - \Phi_2)\, dS \tag{2.5}$$

Denoting the distance between S_1 and S_2 by r_0, for $D \ll r_0$, we have

$$\Phi - \Phi_2 = \Phi_1 (\mathbf{u}, \quad \boldsymbol{\omega}, \quad \mathbf{x}) - \frac{\partial v_i}{\partial x_j} (\mathbf{q})\, \varphi_{ij} + O(D^3) \tag{2.6}$$
$$(\mathbf{u} = \mathbf{q}^{\cdot} - \mathbf{v})$$

The subscript 2 at the unperturbed stream velocity is here and subsequently omitted. Functions φ_{ij} are harmonic outside S_1 and satisfy at S_1 the condition

$$\partial \varphi_{ij} / \partial n = \Delta x_j n_i \tag{2.7}$$

In fact, it follows from (1.2) that for $D \ll r_0$ at S_2 the sum $\Phi_1 + \Phi_1'$ is of the order of D^3. Hence from (1.1) and (1.3) we obtain that at S_2

$$\partial \Phi_2' / \partial n = O(D^3)$$

Consequently, $\Phi_2' = O(D^3)$ everywhere, and from (1.1) we have

$$\Phi - \Phi_2 = \Phi_1 + \Phi_1' + O(D^3) \tag{2.8}$$

At S_1 the boundary condition (1.3) yields

$$\partial \Phi_1' / \partial n = -\partial \Phi_2 / \partial n + O(D^3)$$

Substituting into this equation the first terms of expansion of Φ_2 into a Taylor series at point $\mathbf{x} = \mathbf{q}$, we obtain for Φ_1' an expression of the form

$$\Phi_1' = - v_i \varphi_i - \varphi_{ij} \partial v_i / \partial x_j + O(D^3) \tag{2.9}$$

where functions φ_i and φ_{ij}, which are harmonic outside S_1, satisfy at S_1 conditions (2.2) and (2.7). Taking into account (2.1) and substituting (2.9) into (2.8), we obtain (2.6).

The integral (2.5) is determined with the use of (2.6) and (2.1) – (2.4) in terms of functions T_1 and Γ which are found by solving the problem of motion of a body in a boundless fluid, and the contribution of potentials φ_{ij} is calculated by Green's formula.

To within small magnitudes of the order D^5 the Lagrangian function (1.7) assumes the form

$$L = T_1 (\mathbf{u}, \boldsymbol{\omega}) - p(\mathbf{q}, t) V - \rho \Gamma_{ijk} u_i \frac{\partial v_j}{\partial x_k} (\mathbf{q}, t) \qquad (\mathbf{u} = \mathbf{q}^{\cdot} - \mathbf{v}(\mathbf{q}, t)) \tag{2.10}$$

where point q is the center of volume (center of gravity, if the body is homogeneous), $\mathbf{v}(\mathbf{q}, t)$ and $p(\mathbf{q}, t)$ are, respectively, the velocity and the pressure of the stream in the absence of a body, and V is the volume of the body. The first term in (2.10) corresponds to the kinetic energy of a boundless fluid in which the body moves at the relative velocity u.

For a known Lagrangian function (2.10) the problem of determination of hydrodynamic reactions reduces to differentiation. It is important to bear in mind that to obtain the same accuracy of determination of forces and moments by the Lagrangian equations as by the Cauchy-Lagrange integral it is sufficient to take in the expansion of the unperturbed stream potential Φ_2 terms whose magnitude is by a unit lower. The use of Lagrangian equations, thus simplifies the solution of the problem of hydrodynamic reactions on the body in a nonuniform stream.

For $T_1 + T_{12}$ the energy of the hydrodynamic interaction between two bodies is readily calculated by formula (1.6). Using the value of integral (2.5) determined to within D^5 and omitting subscript 2 at the unperturbed velocity, from (1.6) we obtain

$$\frac{T_{12}}{\rho} = v_i\left(\frac{v_j}{2} - q_j^{\cdot}\right)(\lambda_{ij} + V\delta_{ij}) - v_i\omega_j\lambda_{ij+3} - u_i\frac{\partial v_j}{\partial x_k}\Gamma_{ijk} \qquad (2.11)$$

where δ_{ij} is the Kronecker delta. For fairly great distances between bodies we can neglect in (2.11) the terms which are quadratic with respect to the stream velocity $\mathbf{v}$ induced by the second body. If the distance between any two bodies is considerable in comparison with their dimensions, the hydrodynamic interaction energy in a system of N bodies can be determined by the energies $T_{\alpha\beta}$ (α, $\beta = 1,2,\ldots, N$ and $\alpha \neq \beta$) of hydrodynamic interaction between two bodies, which are calculated with the disregard of terms which are quadratic with respect to $\mathbf{v}$. The total kinetic energy of fluid is equal to the sum of kinetic energies T_α of motion of the αth body in a boundless fluid and of the energy of paired interactions $T_{\alpha\beta}$

$$T = \sum_{\alpha=1}^{N} T_\alpha + \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} T_{\alpha\beta} \qquad (T_{\alpha\alpha} = 0) \qquad (2.12)$$

The validity of (2.12) can be proved by induction, by adding to the system of $N - 1$ bodies one more body and calculating the energy of interaction between the latter and the ($N - 1$)-st body on the assumption that in the absence of the $\alpha = N$ body the velocity at point $\mathbf{q}_N$ is approximately equal to the sum of velocities induced by the motion of individual bodies in the absence of all other bodies, an assumption which is valid for fairly great distances between bodies.

The case of a nonrigid body capable of changing its volume and shape is similarly analyzed with the use of general formulas (1.6) and (1.7). The principal terms $T_1 - pV$ which in formula (2.10) are asymptotic with respect to D remain valid for Lagrangian functions, also, in the general case.

3. Forces and moments acting on the solid body. The expression for the Lagrangian function contains components of tensors Γ and λ, which depend on the body position in space. It is convenient to introduce, in addition to the absolute system of coordinates x_i, the system of coordinates z_i attached to the body. The latter is, generally speaking, not inertial. Let y_i denote an inertial system which at the instant t coincides with z_i. If the change from the y_i-axes to the z_i-axes is determined by

$$z_i = e_{ij}(y_j - h_j) \qquad (3.1)$$

then $h_j = 0$ and $e_{ij} = \delta_{ij}$ is valid at instant t . Force $\mathbf{F}$ exerted by the fluid on the body is determined by the Lagrangian equation

$$-\mathbf{F} = \frac{d}{dt}\frac{\partial L}{\partial \mathbf{q}^{\cdot}} - \frac{\partial L}{\partial \mathbf{q}} \qquad (3.2)$$

The determination of the derivative $\partial L / \partial \mathbf{q}$ by (2.10) is elementary, and (3.2) assumes the form (*)

$$-\mathbf{F} = \frac{d\mathbf{Q}}{dt} + V\nabla p + (\mathbf{Q}\nabla)\mathbf{v} + \mathbf{A} \tag{3.3}$$

$$(Q_i = \partial L / \partial q_i^{\cdot}, \ A_i = \rho\Gamma_{ljk}\nabla_i(u_l\nabla_k v_j))$$

where Γ_{ljk} are components of tensor Γ along the z_i-axes attached to the body. In the system of coordinates attached to the body the component of the generalized momentum vector $\mathbf{Q}$ is defined by formula

$$\frac{1}{\rho} Q_i = \lambda_{ij}e_{jk}u_k + \lambda_{i3+j}\omega_j - \Gamma_{ijk}e_{jl}e_{km}\nabla_l v_m \tag{3.4}$$

where the components of tensors λ, Γ, and of vector ω are defined in the z_i-system, and of vectors $\mathbf{u}$, $\mathbf{v}$ and ∇ in the y_i-system. Formula (3.4) is convenient because tensors λ and Γ are defined in the z_i-system, while vectors $\mathbf{u}$ and $\mathbf{v}$, and their derivatives are determined in the inertial system of y_i-coordinates.

Formula (3.3) contains in the inertial system y_i the derivative of momentum with respect to time, which can be determined with the use of

$$d\mathbf{Q} / dt = d'\mathbf{Q} / dt + \omega \times \mathbf{Q}$$

Here and subsequently a prime denotes a derivative with respect to time in the z_i-system. This derivative is readily determined by differentiating (3.4) and taking into account that tensors λ and Γ are independent of time and that at the instant when the y_i and z_i axes coincide

$$\frac{d}{dt} e_{ij} = \varepsilon_{ijk}\omega_k, \qquad e_{ij} = \delta_{ij} \tag{3.5}$$

The first two terms in (3.4) correspond to the momentum in the case of motion in a uniform stream. The last term takes into account the effect of the velocity gradient of the unperturbed stream and is of the order D^4. The first term in (3.3) coincides with the expression for the force in the case of a boundless fluid at rest at infinity [9]. When the second term in (3.3) is taken into account, this formula defines an accelerated uniform stream [5]. The last two terms take into account the stream nonuniformity and, generally speaking, $|(\mathbf{Q}\nabla)\mathbf{v}| \sim D^3$ and $|\mathbf{A}| \sim D^4$.

The equation of moments can be derived by introducing Euler's angles θ, φ, and ψ as generalized coordinates supplementing the three q_i-coordinates. The cosines of angles e_{ij} in (3.1) are then functions of Euler's angles. Euler's angles are introduced in such a way that the moving trihedral is brought from the position in which its axes are collinear with y_i into that in which these coincide with z_i by successive turns about its axes: by angle ψ about axis $i = 3$, by angle θ about axis $i = 1$, and by angle φ about axis $i = 3$ [13].

The moment of forces exerted by the fluid on the body with respect to the z_3-axis by virtue of Lagrangian equations is

$$W_3 = -\frac{d}{\partial t}\frac{\partial L}{\partial \varphi^{\cdot}} + \frac{\partial L}{\partial \varphi} \tag{3.6}$$

*) A similar expression for the force obtained by discarding terms of the order D^4 and, in particular, of the term denoted by $\mathbf{A}$ was indicated by Iakimov, after the joint discussion of formula (3.2) with allowance for (2.10).

From the definition of Euler's angles for the derivatives of coefficients of transformation (3.1) with respect to angle φ we obtain

$$\partial e_{ij} / \partial \varphi = \varepsilon_{3ik} e_{kj} \tag{3.7}$$

Taking into consideration the dependence of angular velocity projection on the z_i-axis attached to the body on angle φ and on $\varphi^{\cdot}$, we obtain

$$\partial \omega_i / \partial \varphi = \varepsilon_{3ij} \omega_j, \qquad \partial \omega_i / \partial \varphi^{\cdot} = \delta_{3i} \tag{3.8}$$

When Euler's angles are zero, systems y_i and z_i coincide and a small variation of these does not determine all close positions of the tetrahedrons, since the angle between the z_1-axis and the plane $y_3 = 0$ and that between the z_2-axis and the plane $y_1 = 0$ are magnitudes of the second order of smallness. For Euler's angles of the first order of smallness, solid body turns about an axis which does not lie in the plane $y_2 = 0$ are indeterminate. Owing to this, θ, φ and ψ at this point cannot be taken as generalized coordinates. It is then necessary to write the Lagrangian equation at the point θ, φ, and $\psi \neq 0$ and to pass in the final expression for the moment of force W_3 to the limit θ, φ, and $\psi = 0$. With the use of this procedure from (3.5) – (3.8) we obtain

$$\begin{aligned} W_3 &= -\frac{dK_3}{dt} - \varepsilon_{3ij} [\omega_i K_j + u_i Q_j + \rho u_l (\Gamma_{lik} + \Gamma_{lki}) \nabla_k v_j] \\ K_i &= \frac{\partial L}{\partial \omega_i} = \rho \lambda_{3+i\,3+j} \omega_j + \rho \lambda_{3+ij} e_{jk} u_k \\ &\left(\frac{1}{\rho} \frac{dK_3}{dt} = \lambda_{6\,k+3} \frac{d\omega_k}{dt} + \lambda_{6k} \left(\frac{du_k}{dt} - \varepsilon_{kij} \omega_i u_j \right) \right) \end{aligned} \tag{3.9}$$

where v_i and u_i are vector components along the y_i-axes whose direction in space is fixed, and ω_i are angular velocity components along the z_i-axes attached to the body. Expressions for W_1 and W_2 are obtained from (3.9) by a cyclic permutation of subscripts 1, 2, 3. Finally, for the moment $\mathbf{W}$ exerted by the fuid on the body we can write

$$-\mathbf{W} = \frac{d'\mathbf{K}}{dt} + \boldsymbol{\omega} \times \mathbf{K} + \mathbf{u} \times \mathbf{Q} + \mathbf{B} \tag{3.10}$$

$$(B_i = \rho u_l (\Gamma_{ljk} + \Gamma_{lkj}) \varepsilon_{ikm} \nabla_m v_j)$$

The form of (3.10) differs from that for the moment in the case of a boundless fluid at rest at infinity [9] only by the presence of the additional term $|\mathbf{B}| \sim D^4$ and of term of the same order in $\mathbf{Q}$ in (3.4).

The principal terms in (3.3), (3.4) and (3.9) – (3.10) are of the order D^3. The derived formulas are valid to within small magnitudes of order D^5, if coordinates $\dot{q}_i$ are those of the center of volume. The moment at any other point is determined by adding to the obtained value the moment of force $\mathbf{F}$ about that point.

4. The symmetric body. Let a body be symmetric with respect to three mutually perpendicular planes. Taking the latter as the coordinate planes of system z_i, from (2.1) – (2.4) and (2.5), we obtain

$$\Gamma_{ikj} = 0; \qquad \lambda_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \tag{4.1}$$

Using (3.3) and (3.4) and allowing for (4.1), for the first component of force $\mathbf{F}_1$ we can write (*)

$$\frac{F_1}{\rho} = -\lambda_{11}q_1^{\cdot\cdot} - (V + \lambda_{11})\frac{1}{\rho}\frac{\partial p}{\partial y_1} + (\lambda_{11} - \lambda_{22})(q_2^{\cdot} - v_2) \times$$
$$\left(\frac{\partial v_1}{\partial y_2} - \omega_3\right) + (\lambda_{11} - \lambda_{33})(q_3^{\cdot} - v_3)\left(\frac{\partial v_1}{\partial y_3} + \omega_2\right) + \lambda_{11}\frac{\partial U}{\partial y_1} \qquad (4.2)$$

If $\lambda_{11} = \lambda_{22} = \lambda_{33} = \lambda$, which, for example, is valid for a sphere and a cube, formula (4.2) reduces to a particularly simple form. The motion of a solid body (of such form) in a nonuniform stream is the same as that of a material particle in a force field whose force function is $\rho\lambda U - (\lambda + V)\, p$, and, if the nonuniform stream is steady, the equations of motion of the body in it are integrable. If the body is at rest, then, as implied by (4.2),

$$F_1 = \rho\lambda_{11}\left(\frac{\partial v_1}{\partial t} + \nabla_1 U\right) + \nabla_1\left[\frac{\rho}{2}(\lambda_{11}v_1^2 + \lambda_{22}v_2^2 + \lambda_{33}v_3^2) - Vp\right]$$

In the case of a symmetric body formula (3.9) for the moment is, also, considerably simplified by taking into account (4.1)

$$\frac{W_3}{\rho} = -\lambda_{66}\frac{d\omega_3}{dt} + (\lambda_{11} - \lambda_{22})\, u_1 u_2 + (\lambda_{44} - \lambda_{55})\,\omega_1\omega_2 \qquad (4.3)$$

Similar expressions are valid for W_2 and W_1.

An important conclusion follows from formula (4.3), viz., that the stream nonuniformity does not affect the rotation of the body (with an accuracy to within D^5), provided the body has three mutually orthogonal planes of symmetry.

It will be readily seen that in the case of a body moving in a steady uniform stream, formulas (4.2) and (4.3) agree with the known formulas for forces and moments acting on a symmetric body moving in a perfect incompressible fluid [14].

The derived theory of hydrodynamic interaction between bodies is, also, valid in the plane case, if circulation around these is absent. The Lagrangian function (2.10) and the ensuing general equations for forces and moments remain valid but are considerably simplified, owing to the reduction of the number of generalized coordinates from six to three. In the particular case of an elliptic or circular cylinder the derived formulas are in agreement with the results presented in [6, 7].

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REFERENCES

1. Kelvin, W.T. and Tait, P., Treatise on Natural Philosophy, Oxford, 1867.
2. Lamb, H., On the forces experienced by a solid moving in an infinite mass of liquid. Quart. J. Math., Vol.19, 1883.
3. Steklov, V.A., On the Motion of a Solid Body in a Fluid, Khar'kov, 1893.
4. Zhukovskii, N.E., A generalization of the Bjerknes problem of hydrodynamic forces acting on pulsating or oscillating bodies in a fluid mass. Collection: Hydrodynamics, Vol.2, Gostekhizdat, Moscow-Leningrad, 1949.

*) The case of a sphere was independently calculated by one of the authors by direct intergration of pressure over it. The result was found to be in complete agreement with (4.2).

5. Khaskind, M.D., Unsteady motion of a rigid body in an accelerating flow of an infinite fluid. PMM Vol. 20, №1, 1956.
6. Gurevich, M.I., Aerodynamic effect of a train on a small body. Izv. Akad. Nauk SSSR, MZhG, №3, 1968.
7. Iakimov, Iu.L., The motion of a cylinder in an arbitrary plane flow of a perfect incompressible fluid. Izv. Akad. Nauk SSSR, MZhG, №2, 1970.
8. Kostiukov, A.A., Interaction between bodies moving in a fluid. Sudostroenie, Leningrad, 1972.
9. Sedov, L.I., Mechanics of Continuous Medium. Vol. 2, "Nauka", Moscow, 1970.
10. Milne-Thomson, L.M., Theoretical Hydrodynamics, 6th ed. Macmillan, 1968.
11. Lamb, H., Hydrodynamics. Cambridge University Press, 1953.
12. Serrin, J., Mathematical Foundations of Classical Mechanics of Fluids. Izd. Inostr. Lit., Moscow, 1963.
13. Appel, P.E., Traité de Mécanique Rationelle, Vol. 2, Dynamique des Systèmes, 1893.
14. Kochin, N.E., Kibel, I.A. and Roze, N.V., Theoretical Hydrodynamics, pt. 1. Fizmatgiz, Moscow, 1963.

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ON THE ASYMPTOTIC BEHAVIOR OF A STEADY FLOW OF VISCOUS FLUID AT SOME DISTANCE FROM AN IMMERSED BODY

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The steady flow of a viscous incompressible fluid past a body of finite dimensions is considered. It is assumed that the velocity vector u satisfies condition

$$u - u_\infty = O(R^{-\alpha})$$

where u_∞ is the velocity vector of the oncoming stream, R is the distance from a fixed point of the body, and $\alpha > 1/2$. Terms defining the asymptotic behavior of velocity of the order of $O(R^{-1})$ and $O(R^{-3/2})$ are determined and an estimate of the residual term is given. The derived asymptotic formula for the velocity vortex shows that outside the wake the vortex decreases according to an exponential law.

1. Lemmas. 1.1. Let us consider the steady flow of a viscous incompressible fluid past a body such that $B \subset R^3$. We denote the dimensionless velocity vector and pressure by u and p, respectively. Let $S = \partial B$ be a surface which satisfies the Liapunov conditions. We locate the coordinate origin inside B and select the direction of coordinate axes and the scale so that the oncoming stream velocity u_∞ is (1, 0, 0) and the diameter B is unity.

The steady motion of a viscous fluid is defined by the system of equations

$$u \cdot \nabla u + \operatorname{grad} p = \Delta u / 2\lambda, \qquad \operatorname{div} u = 0 \tag{1.1}$$